The Algebraic Regulator Problem from the State-Space Point of View

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ABSTRACT

We study the algebraic aspects of the regulator problem, using some new ideas in the state-space ("geometric") approach to feedback design problems for linear multivariable systems. Necessary and sufficient conditions are given for the solvability of a general version of this problem, requiring output stability, internal stability, and disturbance decoupling as well. An algorithm is given by which these conditions can be verified from the system parameters.

1. INTRODUCTION

The problem of making a given system follow a certain signal in the presence of disturbances is, of course, a basic one in controller design. Several versions have been under study since the very beginnings of control theory. In recent years, much attention has been paid to the underlying algebraic structure of the problem. The central issue here is to decide on solvability or nonsolvability of the problem for a given set of parameters. Of course, in practice the parameters are not known precisely, and the yes-or-no answer which comes from the algebraic analysis is related in a nontrivial way to the hard-easy scale that is much more familiar to the engineer. Still, we may expect that a good understanding of the cases in which the problem is not solvable will be of help in identifying the crucial features of those control problems that should be classified as "intrinsically difficult." Moreover, if the answer to the algebraic problem is constructive in the sense that it provides an algorithm to find a solution if one exists, then this algorithm may also be

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used as a starting point for the development of software that would be applicable to an extensive class of systems.

Among other factors, these considerations have played a role in the development of several different approaches to (what we shall call) the algebraic regulator problem. State-space methods were used in [1–7], resulting in a constructive solution for a fairly general version of the problem. It was felt, however, that a solution in terms of transfer functions would provide a better starting point for investigations involving (small) parameter changes, and this was one of the incentives for a number of papers using techniques like coprime factorization of transfer matrices [8–16]. The solvability conditions obtained, however, are in part unattractive from the numerical point of view (cf. the conclusions of [15]). Very recently, a new frequency-domain solution has been given in [29].

The purpose of the present paper is to restate the case for the state-space approach. We shall use some new ideas to obtain a constructive solution for a general version of the regulator problem, involving output stability, internal stability, and disturbance decoupling. The main feature of the approach adopted here is that it incorporates (dynamic) observation feedback in a natural way. (The intricacy of working with observation feedback in earlier state-space treatments has sometimes been mentioned as a reason to prefer transfer-matrix techniques: see [10].) We shall give several equivalent formulations of the main result, among which there will be an explicit matrix version that could be a starting point for calculations. This paper improves on the results in [18]. The organization of the paper is as follows. After having introduced some notation and preliminaries in Section 2, we motivate our formulation of the regulator problem in Section 3. Section 4 contains necessary conditions for this problem to be solvable. These conditions are shown to be also sufficient in Section 5, and hence we obtain our basic result. In Section 6, we show that this result leads to a completely constructive solvability criterion. The "internal model principle" is briefly discussed in Section 7, and conclusions follow in Section 8. An appendix is added in which it is shown that the problem considered here, when stripped of its "disturbance decoupling" aspect, is identical to the one discussed in [1] (see also [2, Chapter 7]).

2. NOTATION AND PRELIMINARIES

We shall consider only linear, finite-dimensional systems over $\mathbb{R}$. In general, vector spaces will be indicated by script capitals, linear mappings by Latin capitals, and vectors by lowercase letters. Further conventions in the use of letters are as follows. The generic description for a system is

$$x'(t) = Ax(t) + Bu(t) + Eq(t), \qquad x(t) \in \mathcal{X}, \quad u(t) \in \mathcal{U}, \tag{2.1}$$

$$y(t) = Cx(t), \qquad y(t) \in \mathcal{Y}, \tag{2.2}$$

$$z(t) = Dx(t), \qquad z(t) \in \mathcal{Z}, \tag{2.3}$$

Here, $x(t)$ is called the *state* of the system at time t, $u(t)$ is the *input*, $q(t)$ the *disturbance*, $y(t)$ the *observation*, $z(t)$ the *output*. Our controllers will be devices that produce a control function $u(t)$ from an observation function $y(t)$ in the following way:

$$w'(t) = A_c w(t) + G_c y(t), \qquad w(t) \in \mathcal{W}, \tag{2.4}$$

$$u(t) = F_c w(t) + Ky(t). \tag{2.5}$$

This is called a *compensator*; $w(t)$ is the *compensator state*, and $\mathcal{W}$ is the *compensator state space*. We can combine the equations (2.1–3) and (2.4–5) to form the *extended system*:

$$\frac{d}{dt}\begin{pmatrix} x \\ w \end{pmatrix}(t) = \begin{pmatrix} A + BKC & BF_c \\ G_c C & A_c \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix}(t) + \begin{pmatrix} E \\ 0 \end{pmatrix} q(t), \tag{2.6}$$

$$z(t) = \begin{pmatrix} D & 0 \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix}(t). \tag{2.7}$$

We denote

$$A_e = \begin{pmatrix} A + BKC & BF_c \\ G_c C & A_c \end{pmatrix}, \tag{2.8}$$

and call this the *extended system matrix*. This mapping acts on the *extended state space* $\mathcal{X}^e := \mathcal{X} \oplus \mathcal{W}$. There are two natural mappings between $\mathcal{X}^e$ and $\mathcal{X}$: the *natural projection* $P: \mathcal{X}^e \to \mathcal{X}$, defined by

$$P\begin{pmatrix} x \\ w \end{pmatrix} = x, \tag{2.9}$$

and the *canonical imbedding* $Q: \mathcal{X} \to \mathcal{X}^e$, defined by

$$Qx = \begin{pmatrix} x \\ 0 \end{pmatrix}. \tag{2.10}$$

A typical form of a control problem is now: given the system (2.1–3), find a compensator of the form (2.4–5) such that the closed-loop system (2.6–7) has certain properties. For the algebraic regulator problem, these properties can be specified in terms of invariant subspaces of the extended system matrix. We shall denote the "bad subspace" of A_e by $\mathcal{X}_b^e(A_e)$, so

$$\mathcal{X}_b^e(A_e) = \sum_{\operatorname{Re}\lambda \geq 0} \sum_{n \in \mathbb{N}} \ker(\lambda I - A_e)^n. \tag{2.11}$$

This subspace of $\mathcal{X}^e$ contains the "unstable modes" of A_e, i.e., the eigendirections corresponding to nondecreasing solutions. We say that we have *output stability* in the closed-loop system if

$$\mathcal{X}_b^e(A_e) \subset \ker(D \quad 0). \tag{2.12}$$

This means that the output $z(t)$ will converge to zero if no external disturbance is present $[q(t) = 0]$. Another property of interest is *disturbance decoupling*: we say that the closed-loop system has this property if there exists an A_e-invariant subspace $\mathcal{M}$ such that

$$\operatorname{im}\begin{pmatrix} E \\ 0 \end{pmatrix} \subset \mathcal{M} \subset \ker(D \quad 0). \tag{2.13}$$

This means that the behavior of $z(t)$ is completely unaffected by that of $q(t)$. If we have both output stability and disturbance decoupling, then the output $z(t)$ converges to zero regardless of the behavior of $q(t)$. Note that these properties can also be formulated in terms of subspaces of $\mathcal{X}$: (2.12) is equivalent to

$$P\mathcal{X}_b^e(A_e) \subset \ker D, \tag{2.14}$$

and (2.13) is the same as

$$\operatorname{im} E \subset Q^{-1}\mathcal{M} \subset P\mathcal{M} \subset \ker D. \tag{2.15}$$

A third property will be discussed below.

For a while, let us concentrate on the pair (A, B) of *system mapping* and *input mapping* [see (2.1)]. A subspace $\mathcal{V}$ of $\mathcal{X}$ is said to be (A, B)*-invariant* if there exists a "state feedback mapping" $F: \mathcal{X} \to \mathcal{U}$ such that $\mathcal{V}$ is $(A + BF)$-invariant. If $\mathcal{V}$ is (A, B)-invariant, the set of all mappings $F: \mathcal{X} \to \mathcal{U}$ such that

$(A+BF)\mathcal{V} \subset \mathcal{V}$ is denoted by $\mathbf{F}(\mathcal{V})$. An alternative characterization of (A, B)-invariance can be given as follows [2, Lemma 4.2]:

LEMMA 2.1. *A subspace $\mathcal{V}$ of $\mathcal{X}$ is (A, B)-invariant if and only if*

$$A\mathcal{V} \subset \mathcal{V} + \operatorname{im} B. \tag{2.16}$$

From this, it is easily seen that the set of (A, B)-invariant subspaces is closed under subspace addition. Consequently, the set of (A, B)-invariant subspaces that are contained in a given subspace $\mathcal{K}$ (which set is never empty, because the zero subspace is (A, B)-invariant) has a unique largest element which is denoted by $\mathcal{V}^*(\mathcal{K})$. An algorithm to construct $\mathcal{V}^*(\mathcal{K})$ for any given $\mathcal{K}$ can be found in [2, p. 91].

Given an (A, B)-invariant subspace $\mathcal{V}$, it will be important for us to know how the eigenvalues of $A+BF$ can be manipulated when F may be chosen from the class $\mathbf{F}(\mathcal{V})$. To describe the situation, it is convenient to introduce the following notation. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are invariant subspaces for some linear mapping T, and $\mathcal{L}_1 \subset \mathcal{L}_2$, then $T: \mathcal{L}_2/\mathcal{L}_1$ will denote the factor mapping induced on the quotient space $\mathcal{L}_2/\mathcal{L}_1$ by the restriction of T to $\mathcal{L}_2$. In matrix terms, this simply means that if the matrix of T can be written, with respect to a suitable basis, in the block form

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{22} & T_{23} \\ 0 & 0 & T_{33} \end{pmatrix}, \tag{2.17}$$

then the matrix of $T: \mathcal{L}_2/\mathcal{L}_1$ is T_{22}. If $\mathcal{L}_1 = \{0\}$, we shall write $T: \mathcal{L}_2$ instead of $T: \mathcal{L}_2/\{0\}$. We can now formulate the following result ([24]; see also [2, Corollary 5.2, Theorem 4.4]):

LEMMA 2.2. *Let $\mathcal{V}$ be an (A, B)-invariant subspace. Then the smallest $(A+BF)$-invariant subspace containing $\operatorname{im} B \cap \mathcal{V}$ is the same for all $F \in \mathbf{F}(\mathcal{V})$. Denote this subspace by $\mathcal{R}$, and let $\mathcal{S}$ be the smallest A-invariant subspace containing both $\operatorname{im} B$ and $\mathcal{V}$. Then $\mathcal{S}$ is $(A+BF)$-invariant for all F, and we have for all $F_1, F_2 \in \mathbf{F}(\mathcal{V})$*

$$A+BF_1: \mathcal{X}/\mathcal{S} = A: \mathcal{X}/\mathcal{S}, \tag{2.18}$$

$$A+BF_1: \mathcal{V}/\mathcal{R} = A+BF_2: \mathcal{V}/\mathcal{R}. \tag{2.19}$$

Moreover, for any real polynomials $p_1(\lambda)$ and $p_2(\lambda)$ with $\deg(p_1) = \dim \mathcal{S} -$

$\dim \mathcal{V}$ *and* $\deg(p_2) = \dim \mathcal{R}$, *there exists an* $F \in \mathbf{F}(\mathcal{V})$ *such that the characteristic polynomials of* $A + BF: \mathcal{S}/\mathcal{V}$ *and* $A + BF: \mathcal{R}$ *are equal to* $p_1(\lambda)$ *and* $p_2(\lambda)$, *respectively.*

The content of this lemma can conveniently be expressed in the form of a diagram, in which the words "free" and "fixed" refer to the eigenvalues of $A + BF$ when F may be chosen from $\mathbf{F}(\mathcal{V})$:

$$\begin{array}{ll} & \mathcal{X} \\ \text{fixed} & \\ & \mathcal{S} \\ \text{free} & \\ & \mathcal{V} \\ \text{fixed} & \\ & \mathcal{R} \\ \text{free} & \\ & \{0\} \\ A + BF & \\ [F \in \mathbf{F}(\mathcal{V})] & \end{array} \tag{2.20}$$

An (A, B)-invariant subspace $\mathcal{V}$ is called a *controllability subspace* if $\sigma(A + BF: \mathcal{V})$ is free [2, p. 102], and it is called *strongly invariant* if $\sigma(A + BF: \mathcal{X}/\mathcal{V})$ is fixed. If there exists an $F \in \mathbf{F}(\mathcal{V})$ such that $\sigma(A + BF: \mathcal{V}) \subset \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda < 0\}$, then $\mathcal{V}$ is called a *stabilizability subspace*.

For brevity of notation, let us write

$$\mathbb{C}_g = \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda < 0\}, \qquad \mathbb{C}_b = \mathbb{C} \setminus \mathbb{C}_g. \tag{2.21}$$

(Other partitionings of the complex plane may be used, for instance to express stronger stability requirements. The effects on the theory will be none, provided that the partitioning is symmetric with respect to the real axis, and $\mathbb{C}_g \cap \mathbb{R} \neq \varnothing$.) We have already introduced $\mathcal{X}_b^e(A_e)$, and the notation $\mathcal{X}_g^e(A_e), \mathcal{X}_g(A), \mathcal{X}_b(A)$, etc. will refer in an obvious way to the modal subspaces corresponding to the part of $\mathbb{C}$ indicated by the subscript. For any subspace $\mathcal{L}$, we use the following notation for the smallest A-invariant subspace containing $\mathcal{L}$ and for the largest A-invariant subspace contained in $\mathcal{L}$:

$$\langle A | \mathcal{L} \rangle := \sum_{k \in \mathbb{Z}_+} A^k \mathcal{L}, \tag{2.22}$$

$$\langle \mathcal{L} | A \rangle := \bigcap_{k \in \mathbb{Z}_+} A^{-k} \mathcal{L}. \tag{2.23}$$

A strongly invariant subspace of particular interest is

$$\mathcal{X}_{\text{stab}} := \mathcal{X}_g(A) + \langle A | \operatorname{im} B \rangle, \tag{2.24}$$

which is easily seen to be the largest stabilizability subspace in $\mathcal{X}$. More generally, one can prove [19, p. 26; 2, p. 114] that the set of all stabilizability subspaces contained in a given subspace $\mathcal{K}$ has a unique largest element, which will be denoted by $\mathcal{V}_g^*(\mathcal{K})$. Let $\mathcal{V}$ be an (A, B)-invariant subspace. It is seen from Lemma 2.2 that there exists $F \in \mathbf{F}(\mathcal{V})$ such that $\sigma(A + BF: \mathcal{X}/\mathcal{V}) \subset \mathbf{C}_g$ if and only if $\mathcal{S} + \mathcal{X}_g(A) = \mathcal{X}$, where $\mathcal{S} = \langle A | \operatorname{im} B + \mathcal{V} \rangle$. In this case, we shall say that $\mathcal{V}$ is *outer-stabilizable*. It is easily proved that $\langle A | \operatorname{im} B + \mathcal{V} \rangle = \langle A | \operatorname{im} B \rangle + \mathcal{V}$, and so we obtain the following characterization of outer-stabilizability.

LEMMA 2.3. *An (A, B)-invariant subspace $\mathcal{V}$ is outer-stabilizable if and only if*

$$\mathcal{V} + \mathcal{X}_{\text{stab}} = \mathcal{X}. \tag{2.25}$$

Everything that has been said above about the pair (A, B) can be dualized to statements about the pair (C, A) of *output mapping* and *state mapping*. We shall quickly go through the most important notions. A subspace $\mathcal{T}$ of $\mathcal{X}$ is said to be (C, A)*-invariant* if there exists a mapping $\mathcal{G}: \mathcal{Y} \to \mathcal{X}$ such that $\mathcal{T}$ is $(A - GC)$-invariant, or, equivalently, if

$$A(\mathcal{T} \cap \ker C) \subset T. \tag{2.26}$$

The set of all mappings $G: \mathcal{Y} \to \mathcal{X}$ such that $(A - GC)\mathcal{T} \subset \mathcal{T}$ is denoted by $\mathbf{G}(\mathcal{T})$. A (C, A)-invariant subspace $\mathcal{T}$ is said to be a *detectability subspace* if there exists $G \in \mathbf{G}(\mathcal{T})$ such that $\sigma(A - GC: \mathcal{X}/\mathcal{T}) \subset \mathbf{C}_g$. For every subspace $\mathcal{E}$, there is a smallest detectability subspace containing it, which will be denoted by $\mathcal{T}_g^*(\mathcal{E})$. We define

$$\mathcal{X}_{\text{det}} := \mathcal{T}_g^*(\{0\}) = \mathcal{X}_b(A) \cap \langle \ker C | A \rangle. \tag{2.27}$$

This is the smallest subspace modulo which the state can be detected when all inputs are zero.

We now return to the specification of properties for the closed-loop system (2.6–7). It is easily seen that the subspace $Q\mathcal{X}_{\text{det}}$ is always A_e-

invariant, and that $A:\mathcal{X}_{\rm det}$ is similar to $A_e: Q\mathcal{X}_{\rm det}$. The subspace $P^{-1}(\mathcal{X}_{\rm det} + \mathcal{X}_{\rm stab})$ is also always A_e-invariant, and $A_e: \mathcal{X}^e/P^{-1}(\mathcal{X}_{\rm det} + \mathcal{X}_{\rm stab})$ is similar to $A: \mathcal{X}/\mathcal{X}_{\rm det} + \mathcal{X}_{\rm stab})$. This leads immediately to the following result.

LEMMA 2.4. *For any compensator of the form* (2.4–5) *applied to the system* (2.1–3), *the extended system matrix A_e given by* (2.8) *will satisfy*

$$\dim \mathcal{X}_b^e(A_e) \geqslant \dim \mathcal{X}_{\rm det} + \operatorname{codim}(\mathcal{X}_{\rm det} + \mathcal{X}_{\rm stab}). \tag{2.28}$$

We shall say that the closed-loop system (2.6–7) is *internally stable* if equality holds in (2.28). This nomenclature will be explained in the next section.

Finally, we shall need a concept that is related to the triple A, B, and C. A (C, A, B)*-pair* [17] is an (ordered) pair of subspaces $(\mathcal{T}, \mathcal{V})$ in which $\mathcal{T}$ is (C, A)-invariant, $\mathcal{V}$ is (A, B)-invariant, and $\mathcal{T} \subset \mathcal{V}$. The following result [18, Lemma 4.2] will be instrumental.

LEMMA 2.5. *Let $(\mathcal{T}, \mathcal{V})$ be a (C, A, B)-pair. Then there exists a mapping $K: \mathcal{Y} \to \mathcal{U}$ such that $(A + BKC)\mathcal{T} \subset \mathcal{V}$.*

This means that in situations where we are allowed to replace A by $A + BKC$ (applying a preliminary static output feedback), it is no restriction of generality to assume that $A\mathcal{T} \subset \mathcal{V}$. Note that the properties we discussed above for the pair (A, B) are all feedback invariant: they would have been the same for any pair of the form $(A + BF, B)$. Likewise, the properties relating to the pair (C, A) would have been the same for any pair of the form $(C, A - GC)$. Consequently, the change from A to $A + BKC$ changed neither the input-to-state nor the state-to-output structure, which makes it a transformation that is applicable under many circumstances. If we have to do with several (C, A, B)-pairs $(\mathcal{T}_i, \mathcal{V}_i)$ $(i = 1, \ldots, k)$, there does not necessarily exist a K such that $(A + BKC)\mathcal{T}_i \subset \mathcal{V}_i$ for all i; we shall say that the pairs $(\mathcal{T}_i, \mathcal{V}_i)$ are *compatible* if such a K does exist.

3. PROBLEM STATEMENT

A common control setup for a plant to follow a reference signal in the face of disturbances is depicted in Figure 1. Here, the prefilter, the precompensator and the feedback compensator are elements that are to be constructed by the designer in such a way that the error will tend to zero for every choice of

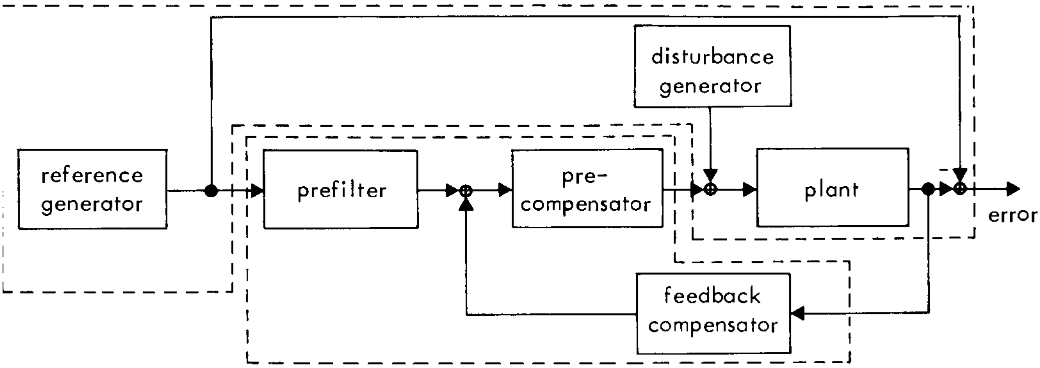

FIG. 1. Control scheme.

initial conditions in the reference generator, the disturbance generator, and the plant. The diagram can be reorganized to display more clearly the interface between the given elements and the elements that are to be constructed, as shown in Figure 2.

The scheme can be simplified and generalized at the same time, as shown in Figure 3. All the given elements have been taken together under the name "system," and the control elements are represented by one feedback processor called the "compensator." Also, an additional external disturbance has been added for which no knowledge of the dynamics is assumed. (This may be quite natural, for instance, when this disturbance is used to model a lack of information about certain system parameters.) The error has been renamed as

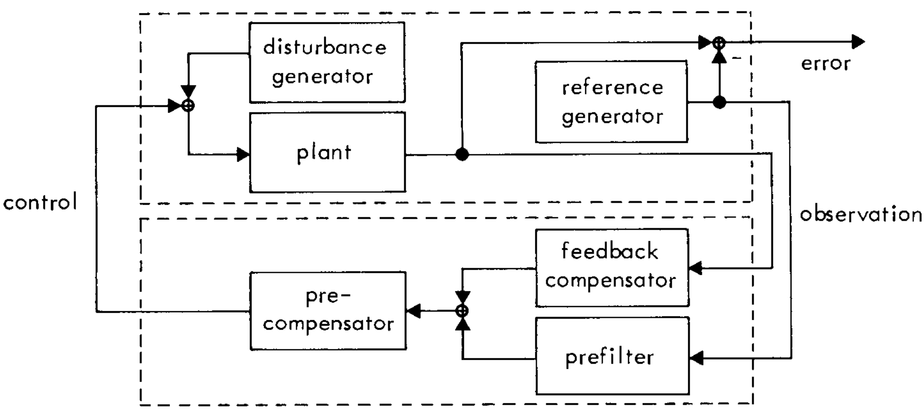

FIG. 2. Reorganized control scheme.

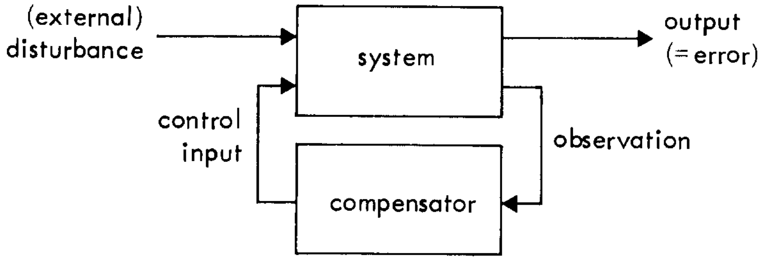

FIG. 3. Simplified and generalized control scheme.

simply "output"; the longer term "variables-to-be-controlled" is also sometimes used.

We are now in the situation described in the previous section. The system is described by the equations (2.1–3), the compensator equations are given by (2.4–5), and the closed-loop system as a whole is described by (2.6–7). The question is, of course, whether we are still able to properly define our control objectives in the present context, in which the distinction between plant, disturbance, and reference has seemingly disappeared.

To answer this question, we break down the system mapping A using the chain of invariant subspaces $\{0\} \subset \mathcal{X}_{\text{det}} \subset \mathcal{X}_{\text{det}} + \mathcal{X}_{\text{stab}} \subset \mathcal{X}$. Taking into account the facts that $\mathcal{X}_{\text{det}} \subset \ker C$ and that $\operatorname{im} B \subset \mathcal{X}_{\text{det}} + \mathcal{X}_{\text{stab}}$, this enables us to rewrite the equations

$$x'(t) = Ax(t) + Bu(t), \tag{3.1}$$

$$y(t) = Cx(t) \tag{3.2}$$

in the following way:

$$x_1'(t) = A_{11}x_1(t) + A_{12}x_2(t) + A_{13}x_3(t) + B_1u(t), \tag{3.3}$$

$$x_2'(t) = A_{22}x_2(t) + A_{23}x_3(t) + B_2u(t), \tag{3.4}$$

$$x_3'(t) = A_{33}x_3(t), \tag{3.5}$$

$$y(t) = C_2x_2(t) + C_3x_3(t). \tag{3.6}$$

Pictorially, we have the diagram in Figure 4. This makes it natural to interpret $x_1(t)$ (corresponding to $A : \mathcal{X}_{\text{det}}$) as representing irrelevant plant variables. That is, we assume that we are not in the fundamentally hopeless situation in which there are unobservable unstable relevant plant modes. The vector $x_3(t)$ is naturally interpreted as representing the state variables of the reference and (internal) disturbance generators. Again, supposing that $x_3(t)$ partly represents plant variables would bring us into a fundamentally wrong situation, this time because of the presence of unstable uncontrollable plant modes. It can be argued (see for instance [7]) that it is reasonable to assume that $\mathcal{X}_{\text{det}} = \{0\}$, but we shall take the option of performing the mathematical analysis in full generality, to see if the outcome agrees with our interpretations.

With this background, it is now reasonable to formulate the following specifications for the closed-loop system. To ensure that the system output (which represents the difference between reference signal and actual plant behavior) will tend to zero in spite of the internal and external disturbances, we ask for *output stability* and *disturbance decoupling* [(2.12) and (2.13)]. Moreover, we want the plant to be stabilized. Using the interpretation discussed above, this requirement is expressed by the condition of *internal stability*:

$$\dim \mathcal{X}_b^e(A_e) = \dim \mathcal{X}_{\text{det}} + \operatorname{codim}(\mathcal{X}_{\text{det}} + \mathcal{X}_{\text{stab}}). \tag{3.7}$$

So the *algebraic regulator problem* that will be discussed in this paper is: Given a system of the form (2.1–3), find necessary and sufficient conditions for the existence of a compensator of the form (2.4–5) such that the closed-loop system (2.6–7) has the properties of output stability, disturbance decoupling, and internal stability; and give an algorithm to construct such a compensator, if one exists. We use the qualifier "algebraic" because this problem does not include issues like sensitivity to parameter changes, the

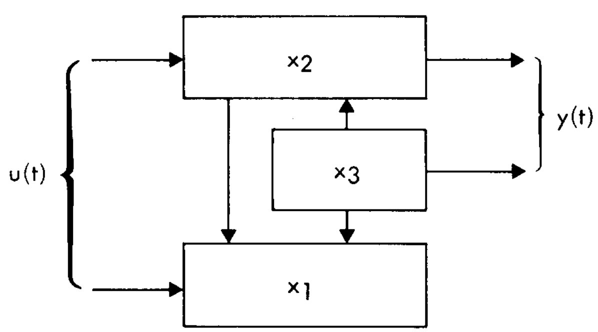

FIG. 4. Decomposition of a general linear system.

response of the system to signals other than those it has been designed for, efficient and numerically stable computational algorithms, and so on. It will be shown in the Appendix that the algebraic regulator problem as it is formulated here is a strict generalization of the problem considered in [1] (also in [2, Chapter 7]).

4. NECESSITY

We start with the following simple but basic observation (cf. [17]).

LEMMA 4.1. *Let A_e be an extended system matrix of the form (2.8), and suppose that $\mathfrak{M}_1,\dots,\mathfrak{M}_k$ are A_e-invariant subspaces. Then the pairs $(Q^{-1}\mathfrak{M}_1, P\mathfrak{M}_1),\dots,(Q^{-1}\mathfrak{M}_k, P\mathfrak{M}_k)$ are compatible (C, A, B)-pairs.*

Proof. Take $i \in \{1,\dots,k\}$, and let $x \in Q^{-1}\mathfrak{M}_i \cap \ker C$. Then

$$\begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathfrak{M}_i,$$

and consequently

$$\begin{pmatrix} A+BKC & BF_c \\ G_cC & A_c \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} Ax \\ 0 \end{pmatrix} \in \mathfrak{M}_i. \tag{4.1}$$

We see that $Ax \in \mathfrak{M}_i$, showing that $Q^{-1}\mathfrak{M}_i$ is (C, A)-invariant. Next, let $x \in P\mathfrak{M}_i$ and take $w \in \mathfrak{W}$ such that

$$\begin{pmatrix} x \\ w \end{pmatrix} \in \mathfrak{M}_i.$$

Then

$$\begin{pmatrix} A+BKC & BF_c \\ G_cC & A_c \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix} = \begin{pmatrix} Ax + B(KCx + F_cw) \\ G_cCx + A_cw \end{pmatrix} \in \mathfrak{M}_i. \tag{4.2}$$

Hence, $Ax + B(KCx + F_cw) \in P\mathfrak{M}_i$, which implies that $Ax \in P\mathfrak{M}_i + \operatorname{im} B$ and that $P\mathfrak{M}_i$ is (A, B)-invariant. Finally, let $x \in Q^{-1}\mathfrak{M}_i$. We have

$$\begin{pmatrix} A+BKC & BF_c \\ G_cC & A_c \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} (A+BKC)x \\ G_cCx \end{pmatrix} \in \mathfrak{M}_i, \tag{4.3}$$

which shows that $(A + BKC)Q^{-1}\mathcal{M}_i \subset P\mathcal{M}_i$. Since K does not depend on i, this completes the proof. ∎

We now want to bring in the aspect of eigenvalue assignment. First, recall the following result, which can be proved by standard means.

LEMMA 4.2. *Let $T: \mathcal{X} \to \mathcal{X}$ be a linear mapping, and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be invariant subspaces for T, with $\mathcal{L}_1 \subset \mathcal{L}_2$. Then, the following are equivalent:*

$$\sigma(T:\mathcal{L}_2/\mathcal{L}_1) \subset C_g, \tag{4.4}$$

$$(\lambda I - T)^{-1}\mathcal{L}_1 \cap \mathcal{L}_2 = \mathcal{L}_1 \qquad \forall \lambda \in \mathbf{C}_b, \tag{4.5}$$

$$(\lambda I - T)\mathcal{L}_2 + \mathcal{L}_1 = \mathcal{L}_2 \qquad \forall \lambda \in \mathbf{C}_b. \tag{4.6}$$

Suppose that $\mathcal{V}_1$ and $\mathcal{V}_2$ are (A, B)-invariant subspaces, and $\mathcal{V}_1 \subset \mathcal{V}_2$. We shall say that (A, B) is *stabilizable between* $\mathcal{V}_1$ *and* $\mathcal{V}_2$ if there exists an $F \in \mathbf{F}(\mathcal{V}_1) \cap \mathbf{F}(\mathcal{V}_2)$ such that $\sigma(A + BF: \mathcal{V}_2/\mathcal{V}_1) \subset \mathbf{C}_g$. We have the following characterization of this property.

LEMMA 4.3. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be (A, B)-invariant subspaces, with $\mathcal{V}_1 \subset \mathcal{V}_2$. Then (A, B) is stabilizable between $\mathcal{V}_1$ and $\mathcal{V}_2$ if and only if*

$$(\lambda I - A)\mathcal{V}_2 + \mathcal{V}_1 + \operatorname{im} B = \mathcal{V}_2 + \operatorname{im} B \qquad \forall \lambda \in \mathbf{C}_b. \tag{4.7}$$

Proof. First, suppose there exists $F \in \mathbf{F}(\mathcal{V}_2) \cap \mathbf{F}(\mathcal{V}_1)$ such that $A + BF: \mathcal{V}_2/\mathcal{V}_1$ is stable. According to Lemma 4.2, we then have

$$[\lambda I - (A + BF)]\mathcal{V}_2 + \mathcal{V}_1 = \mathcal{V}_2 \qquad \forall \lambda \in \mathbf{C}_b. \tag{4.8}$$

Adding $\operatorname{im} B$ on both sides now leads immediately to (4.7), if one uses the obvious equality

$$[\lambda I - (A + BF)]\mathcal{V}_2 + \operatorname{im} B = (\lambda I - A)\mathcal{V}_2 + \operatorname{im} B. \tag{4.9}$$

Next, suppose that (4.7) holds. Construct a mapping $F_0 \in \mathbf{F}(\mathcal{V}_1) \cap \mathbf{F}(\mathcal{V}_2)$ by first defining F_0 on $\mathcal{V}_1$ such that $(A + BF_0)\mathcal{V}_1 \subset \mathcal{V}_1$, then extending F_0 on $\mathcal{V}_2$ in such a way that $(A + BF_0)\mathcal{V}_2 \subset \mathcal{V}_2$, and finally extending F_0 in an arbitrary way to a mapping defined on all of $\mathcal{X}$. Consider the controllability subspace

$$\mathcal{R}_2 := \langle A + BF_0 | \operatorname{im} B \cap \mathcal{V}_2 \rangle \tag{4.10}$$

Define $A_0 := A + BF_0 : \mathcal{R}_2$, and let $R: \mathcal{U} \to \mathcal{U}$ be such that $\operatorname{im} BR = \operatorname{im} B \cap \mathcal{V}_2$. Write $B_0 := BR$. By definition, we have $\langle A_0 | \operatorname{im} B_0 \rangle = \mathcal{R}_2$, and so it follows from Lemma 2.3 that every (A_0, B_0)-invariant subspace of $\mathcal{R}_2$ is outer-stabilizable. In particular, there exists an $F_1 : \mathcal{R}_2 \to \mathcal{U}$ such that $(A_0 + B_0 F_1)(\mathcal{R}_2 \cap \mathcal{V}_1) \subset \mathcal{R}_2 \cap \mathcal{V}_1$ and $\sigma(A_0 + B_0 F_1 : \mathcal{R}_2 / \mathcal{R}_2 \cap \mathcal{V}_1)) \subset \mathbb{C}_g$. The mapping $F_0 + RF_1$, which is defined only on $\mathcal{R}_2$, can be extended to a mapping $F: \mathcal{X} \to \mathcal{U}$ in such a way that $F \in \mathbf{F}(\mathcal{V}_1) \cap \mathbf{F}(\mathcal{V}_2)$. We claim that this mapping F satisfies $\sigma(A + BF : \mathcal{V}_2 / \mathcal{V}_1) \subset \mathbb{C}_g$.

To prove this, first note that $A + BF : (\mathcal{R}_2 + \mathcal{V}_1)/\mathcal{V}_1$ is similar to $A + BF : \mathcal{R}_2 / (\mathcal{R}_2 \cap \mathcal{V}_1) = A_0 + B_0 F_1 : \mathcal{R}_2 / (\mathcal{R}_2 \cap \mathcal{V}_1)$, which is stable by construction. Furthermore, we have given that (4.7) holds, and this implies [using (4.9) again]

$$[\lambda I - (A + BF)]\mathcal{V}_2 + \mathcal{V}_1 + \operatorname{im} B = \mathcal{V}_2 + \operatorname{im} B \qquad \forall \lambda \in \mathbb{C}_b. \quad (4.11)$$

Taking intersections with $\mathcal{V}_2$ on both sides, we get

$$[\lambda I - (A + BF)]\mathcal{V}_2 + \mathcal{V}_1 + (\operatorname{im} B \cap \mathcal{V}_2) = \mathcal{V}_2 \qquad \forall \lambda \in \mathbb{C}_b. \quad (4.12)$$

Because $\operatorname{im} B \cap \mathcal{V}_2 \subset \mathcal{R}_2 \subset \mathcal{V}_2$, this implies

$$[\lambda I - (A + BF)]\mathcal{V}_2 + \mathcal{V}_1 + \mathcal{R}_2 = \mathcal{V}_2 \qquad \forall \lambda \in \mathbb{C}_b, \qquad (4.13)$$

which, by Lemma 4.2, means that $\sigma(A + BF : \mathcal{V}_2 / (\mathcal{V}_1 + \mathcal{R}_2)) \subset \mathbb{C}_g$. The proof is done. ∎

We are going to apply this lemma in the following way.

LEMMA 4.4. *Let A_e be an extended system matrix of the form* (2.8). *If $\mathcal{M}_1$ and $\mathcal{M}_2$ are both A_e-invariant subspaces satisfying $\mathcal{M}_1 \subset \mathcal{M}_2$ and $\sigma(A_e : \mathcal{M}_2 / \mathcal{M}_1) \subset \mathbb{C}_g$, then the pair (A, B) is stabilizable between $P\mathcal{M}_1$ and $P\mathcal{M}_2$.*

Proof. Take $x \in P\mathcal{M}_2$, and let $w \in \mathcal{W}$ be such that

$$\begin{pmatrix} x \\ w \end{pmatrix} \in \mathcal{M}_2.$$

Also, take $\lambda \in \mathbb{C}_b$. By Lemma 4.2, there exist vectors

$$\begin{pmatrix} x_1 \\ w_1 \end{pmatrix} \in \mathcal{M}_1 \quad \text{and} \quad \begin{pmatrix} x_2 \\ w_2 \end{pmatrix} \in \mathcal{M}_2$$

such that

$$\begin{pmatrix} x \\ w \end{pmatrix} = (\lambda I - A_e)\begin{pmatrix} x_2 \\ w_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ w_1 \end{pmatrix}. \tag{4.14}$$

In particular, we get

$$x = (\lambda I - A)x_2 - B(KCx_2 + F_c w_2) + x_1. \tag{4.15}$$

This shows that

$$P\mathfrak{M}_2 \subset (\lambda I - A)P\mathfrak{M}_2 + P\mathfrak{M}_1 + \operatorname{im} B \qquad \forall \lambda \in \mathbf{C}_b. \tag{4.16}$$

By the (A, B)-invariance of $P\mathfrak{M}_2$, this is the same as

$$P\mathfrak{M}_2 + \operatorname{im} B = (\lambda I - A)P\mathfrak{M}_2 + P\mathfrak{M}_1 + \operatorname{im} B \qquad \forall \lambda \in \mathbf{C}_b. \tag{4.17}$$

An application of Lemma 4.3 now gives the desired result. ■

Everything that has been said above about the pair (A, B) can be dualized into statements about the pair (C, A). If $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are (C, A)-invariant subspaces such that $\mathfrak{T}_1 \subset \mathfrak{T}_2$, we shall say that the pair (C, A) is *detectable between* $\mathfrak{T}_1$ *and* $\mathfrak{T}_2$ if there exists a $G \in \mathbf{G}(\mathfrak{T}_1) \cap \mathbf{G}(\mathfrak{T}_2)$ such that $\sigma(A - GC: \mathfrak{T}_2/\mathfrak{T}_1) \subset \mathbf{C}_g$. The following results correspond to Lemma 4.3 and Lemma 4.4, respectively:

LEMMA 4.5. *Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be (C, A)-invariant subspaces, with $\mathfrak{T}_1 \subset \mathfrak{T}_2$. Then (C, A) is detectable between $\mathfrak{T}_1$ and $\mathfrak{T}_2$ if and only if*

$$(\lambda I - A)^{-1}\mathfrak{T}_1 \cap \mathfrak{T}_2 \cap \ker C = \mathfrak{T}_1 \cap \ker C \qquad \forall \lambda \in \mathbf{C}_b. \tag{4.18}$$

LEMMA 4.6. *Let A_e be an extended system matrix of the form* (2.8). *If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are both A_e-invariant subspaces, satisfying $\mathfrak{M}_1 \subset \mathfrak{M}_2$ and $\sigma(A_e: \mathfrak{M}_2/\mathfrak{M}_1) \subset \mathbf{C}_g$, then the pair (C, A) is detectable between $Q^{-1}\mathfrak{M}_1$ and $Q^{-1}\mathfrak{M}_2$.*

It is useful to note the following result, which is a direct consequence of Lemma 4.3.

COROLLARY 4.7. *Suppose that $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$ are (A,B)-invariant subspaces, with $\mathcal{V}_1 \subset \mathcal{V}_2$. If the pair (A,B) is stabilizable between $\mathcal{V}_1$ and $\mathcal{V}_2$, then (A,B) is also stabilizable between $\mathcal{V}_1+\mathcal{V}_3$ and $\mathcal{V}_2+\mathcal{V}_3$.*

After these preparations, it is easy to give an extensive list of necessary conditions for the algebraic regulator problem to be solvable.

PROPOSITION 4.8. *Suppose that the compensator* (2.4–5) *provides a solution to the algebraic regulator problem for the system* (2.1–3), *so there exists an A_e-invariant subspace $\mathcal{M}$ such that* (2.13) *holds and such that $\sigma(A_e:\mathcal{X}^e/\mathcal{M})\subset\mathbf{C}_g$, and moreover the dimensional equality* (3.7) *holds. Write $\mathcal{V}:=P\mathcal{M}$, $\mathcal{T}:=Q^{-1}\mathcal{M}$, $\mathcal{V}_0:=P\mathcal{X}_b^e(A_e)$, and $\mathcal{T}_0:=Q^{-1}\mathcal{X}_b^e(A_e)$. Then the following is true*:

(i) *the pairs $(\mathcal{T}_0,\mathcal{V}_0)$ and $(\mathcal{T},\mathcal{V})$ are compatible (C,A,B)-pairs,*
(ii) *$\mathcal{T}_0\subset\mathcal{T}$ and $\mathcal{V}_0\subset\mathcal{V}$,*
(iii) $\operatorname{im} E\subset\mathcal{T}\subset\mathcal{V}\subset\ker D$,
(iv) *(A,B) is stabilizable between $\mathcal{V}_0$ and $\mathcal{V}$ and between $\mathcal{V}$ and $\mathcal{X}$,*
(v) *(C,A) is detectable between $\mathcal{T}_0$ and $\mathcal{T}$ and between $\mathcal{T}$ and $\mathcal{X}$,*
(vi) $\mathcal{V}_0\cap(\mathcal{X}_{\text{det}}+\mathcal{X}_{\text{stab}})=\mathcal{X}_{\text{det}}$.

Proof. Conditions (i) to (v) follow immediately from, respectively, Lemma 4.1, the fact that $\mathcal{X}_b^e(A_e)\subset\mathcal{M}$, the remark leading to (2.15), Lemma 4.4, and Lemma 4.6. To prove (vi), first note that $\mathcal{T}_0$ is, by (v), a detectability subspace. Therefore, $\mathcal{X}_{\text{det}}\subset\mathcal{T}_0\subset\mathcal{V}_0$ and so we have

$$\mathcal{X}_{\text{det}}\subset\mathcal{V}_0\cap(\mathcal{X}_{\text{det}}+\mathcal{X}_{\text{stab}}). \tag{4.19}$$

From (iv), it follows that $\mathcal{V}_0$ is outer-stabilizable, so that $\mathcal{V}_0+\mathcal{X}_{\text{stab}}=\mathcal{X}$ (Lemma 2.3). Consequently, the following dimensional relations hold:

$$\begin{aligned}\dim[\mathcal{V}_0\cap(\mathcal{X}_{\text{det}}+\mathcal{X}_{\text{stab}})] &= \dim\mathcal{V}_0+\dim(\mathcal{X}_{\text{det}}+\mathcal{X}_{\text{stab}})-\dim\mathcal{X}\\ &= \dim\mathcal{V}_0-\operatorname{codim}(\mathcal{X}_{\text{det}}+\mathcal{X}_{\text{stab}})\\ &\leqslant \dim\mathcal{X}_b^e(A_e)-\operatorname{codim}(\mathcal{X}_{\text{det}}+\mathcal{X}_{\text{stab}})\\ &= \dim\mathcal{X}_{\text{det}}.\end{aligned}$$

This shows that in fact equality holds in (4.19). ∎

The list is not completely economical; for instance, it is easy to see that (ii) already implies that the (C, A, B)-pairs $(\mathcal{T}_0, \mathcal{V}_0)$ and $(\mathcal{T}, \mathcal{V})$ are compatible. The extras have been obtained with little effort, however, and the form of the list is convenient for the next section, where we are going to prove that the conditions given above are also sufficient.

5. SUFFICIENCY; MAIN RESULT

There is a general method of compensator construction, in which it is also possible to keep track of the relation between invariant subspaces in the constructed closed-loop system and certain (C, A, B)-pairs in $\mathcal{X}$. Here, we shall only need the following relatively simple result; more elaborate versions are given in [18, Theorem 4.1] and [19, pp. 63–64]. The proof is basically easy, consisting mainly of using natural isomorphisms between subspaces of $\mathcal{X}$ and of $\mathcal{X}^e$, and can be found in the cited references.

LEMMA 5.1. *Let the system* (2.1–3) *be given. Suppose that we have a* (C, A, B)*-pair* $(\mathcal{T}_c, \mathcal{V}_c)$, *an* $F \in \mathbf{F}(\mathcal{V}_c)$ *such that* $\ker F \supset \mathcal{T}_c$, *and a* $G \in \mathbf{G}(\mathcal{T}_c)$ *such that* $\operatorname{im} G \subset \mathcal{V}_c$. *Then a compensator of the form* (2.4–5) *can be defined as follows: Let* $\mathcal{W}$ *be a real vector space of dimension* $\dim \mathcal{V}_c - \dim \mathcal{T}_c$. *Let* R *be a mapping from* $\mathcal{V}_c$ *onto* $\mathcal{W}$ *such that* $\ker R = \mathcal{T}_c$, *and let* R^+ *be any right inverse of* R. *Set* $K = 0$, $F_c = FR^+$, $G_c = RG$, *and* $A_c = R(A + BF - GC)R^+$. *The extended system matrix that is obtained as*

$$A_e = \begin{pmatrix} A & BFR^+ \\ RGC & R(A + BF - GC)R^+ \end{pmatrix} \tag{5.1}$$

has the following eigenvalues:

$$\sigma(A_e) = \sigma(A + BF: \mathcal{V}_c) \cup \sigma(A - GC: \mathcal{X}/\mathcal{T}_c). \tag{5.2}$$

Moreover, if $(\mathcal{T}, \mathcal{V})$ *is a* (C, A, B)*-pair such that* $A\mathcal{T} \subset \mathcal{V}$, $\mathcal{T}_c \subset \mathcal{T} \subset \mathcal{V} \subset \mathcal{V}_c$, $(A + BF)\mathcal{V} \subset \mathcal{V}$, *and* $(A - GC)\mathcal{T} \subset \mathcal{T}$, *then the subspace* $\mathcal{M}$ *of* $\mathcal{X}$ *defined by*

$$\mathcal{M} := \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \middle| x \in \mathcal{T} \right\} + \left\{ \begin{pmatrix} x \\ Rx \end{pmatrix} \middle| x \in \mathcal{V} \right\} \tag{5.3}$$

is A_e-invariant. The subspace

$$\mathcal{M}_c := \left\{ \begin{pmatrix} x \\ Rx \end{pmatrix} \middle| x \in \mathcal{V}_c \right\} \tag{5.4}$$

is also A_e-invariant, and the following similarity relations hold:

$$A_e : \mathcal{X}^e/(\mathcal{M} + \mathcal{M}_c) \cong A - GC : \mathcal{X}/\mathcal{T} \tag{5.5}$$

$$A_e : (\mathcal{M} + \mathcal{M}_c)/\mathcal{M}_c \cong A_e : \mathcal{M}/(\mathcal{M} \cap \mathcal{M}_c) \cong A - GC : \mathcal{T}/\mathcal{T}_c \tag{5.6}$$

$$A_e : (\mathcal{M} + \mathcal{M}_c)/\mathcal{M} \cong A_e : \mathcal{M}_c/(\mathcal{M} \cap \mathcal{M}_c) \cong A + BF : \mathcal{V}_c/\mathcal{V} \tag{5.7}$$

$$A_e : \mathcal{M} \cap \mathcal{M}_c \cong A + BF : \mathcal{V}. \tag{5.8}$$

Pictorially, the relations (5.5–8) can be described as shown in Figure 5.

In order to translate data on a chain of (A, B)-invariant subspaces into data on a feedback mapping, the following lemma is useful.

LEMMA 5.2. *Suppose we have a chain of (A, B)-invariant subspaces $\{0\} = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_{k-1} \subset \mathcal{V}_k = \mathcal{X}$. Also, let mappings $F_i \in \mathbf{F}(\mathcal{V}_i) \cap \mathbf{F}(\mathcal{V}_{i-1})$ be given, for $i = 1,\ldots,k$. Then there exists a mapping $F \in \mathbf{F}(\mathcal{V}_1) \cap \cdots \cap \mathbf{F}(\mathcal{V}_{k-1})$ such that $A + BF : \mathcal{V}_i/\mathcal{V}_{i-l} = A + BF_i : \mathcal{V}_i/\mathcal{V}_{i-1}$ for all $i \in \{1,\ldots,k\}$.*

Proof. Select basis elements $\{x_1^1,\ldots,x_{n_1}^1, x_1^2,\ldots,x_{n_2}^2,\ldots,x_1^k,\ldots,x_{n_k}^k\}$ such that $\{x_1^1,\ldots,x_{n_1}^1,\ldots,x_1^i,\ldots,x_{n_i}^i\}$ forms a basis for $\mathcal{V}_i$ for all $i \in \{1,\ldots,k\}$. Define F by $Fx_j^i = F_i x_j^i$ $(i = 1,\ldots,k;\ j = 1,\ldots,n_i)$. Then F satisfies the requirements. ∎

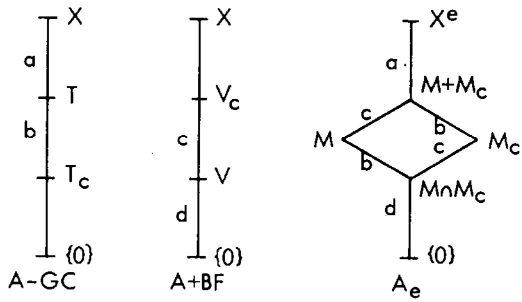

FIG. 5. Regulator construction.

To illustrate the proof, consider the block matrix representations for the mappings $A + BF_i$ $(i = 1,\dots,k)$ and $A + BF$ with respect to the selected basis:

$$
\underset{A+BF_1}{\begin{pmatrix}
* & * & \cdots & * \\
0 & \cdot & & \cdot \\
\vdots & \vdots & & \vdots \\
0 & * & \cdots & *
\end{pmatrix}}
\cdots
\begin{pmatrix}
* & \cdots & * & * & * & \cdots & * \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
* & \cdots & * & \cdot & \cdot & & \cdot \\
0 & \cdots & 0 & * & \cdot & & \cdot \\
\cdot & & \cdot & 0 & \cdot & & \cdot \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & * & \cdots & *
\end{pmatrix}
\cdots
$$

$$
\underset{A+BF_k}{\begin{pmatrix}
* & \cdots & * & * \\
\vdots & & \vdots & \vdots \\
* & \cdots & * & \cdot \\
0 & \cdots & 0 & *
\end{pmatrix}}
\quad
\underset{A+BF}{\begin{pmatrix}
* & \cdots & * & \cdots & * & * \\
0 & & \cdot & & \cdot & \cdot \\
\vdots & & \vdots & & \vdots & \vdots \\
\cdot & & * & & \cdot & \cdot \\
\cdot & & 0 & & \cdot & \cdot \\
\vdots & & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & *
\end{pmatrix}}
$$

It is now not difficult to show that the necessary conditions derived in the previous section are also sufficient.

THEOREM 5.3. *The algebraic regulator problem for the system* (2.1–3) *is solvable if and only if there exist two compatible* (C, A, B)*-pairs* $(\mathcal{T}_0, \mathcal{V}_0)$ *and* $(\mathcal{T}, \mathcal{V})$ *such that*

(i) $\mathcal{T}_0 \subset \mathcal{T}$, $\mathcal{V}_0 \subset \mathcal{V}$,
(ii) $\operatorname{im} E \subset \mathcal{T} \subset \mathcal{V} \subset \ker D$
(iii) (A, B) *is stabilizable between* $\mathcal{V}_0$ *and* $\mathcal{V}$ *and between* $\mathcal{V}$ *and* $\mathcal{X}$,
(iv) (C, A) *is detectable between* $\mathcal{T}_0$ *and* $\mathcal{T}$ *and between* $\mathcal{T}$ *and* $\mathcal{X}$,
(v) $\mathcal{V}_0 \cap (\mathcal{X}_{\mathrm{det}} + \mathcal{X}_{\mathrm{stab}}) = \mathcal{X}_{\mathrm{det}}$.

Proof. In view of Proposition 4.8, it remains to show the sufficiency of the conditions. Employing a preliminary static output feedback if necessary, we may assume that $A\mathcal{T}_0 \subset \mathcal{V}_0$ and $A\mathcal{T} \subset \mathcal{V}$ (see Lemma 2.5 and the remarks following it). It follows that there exists an $F_1 \in \mathbf{F}(\mathcal{V}_0)$ with $\ker F_1 \supset \mathcal{T}_0$. Using (iii) and Lemma 5.2, we see that there exists an $F \in \mathbf{F}(\mathcal{V}_0) \cap \mathbf{F}(\mathcal{V})$ such that $\ker F \supset \mathcal{T}_0$ and $\sigma(A + BF: \mathcal{X}/\mathcal{V}_0) \subset \mathbf{C}_g$. Using (iv) and the dual of Lemma 5.2, we find that there exists a $G \in \mathbf{G}(\mathcal{T}_0) \cap \mathbf{G}(\mathcal{T})$ such that $\sigma(A - GC: \mathcal{X}/\mathcal{T}_0) \subset \mathbf{C}_g$. Now we apply the compensator construction of Lemma 5.1, using $\mathcal{T}_0$ for $\mathcal{T}_c$ and $\mathcal{X}$ for $\mathcal{V}_c$. The invariant subspace $\mathcal{M}$ related to the pair $(\mathcal{T}, \mathcal{V})$ takes care of the disturbance decoupling property, by condition (ii). From the relations (5.5–8), we see that

$$\mathcal{X}_b^e(A_e) \subset \mathcal{M} \cap \mathcal{M}_c = \left\{ \begin{pmatrix} x \\ Rx \end{pmatrix} \middle| x \in \mathcal{V} \right\}.$$

Consequently, $P\mathcal{X}_b^e(A_e) \subset \mathcal{V} \subset \ker D$ and we have output stability. In fact, we see from (5.8) that $\dim \mathcal{X}_b^e(A_e) \leqslant \dim \mathcal{V}_0$. By condition (v), we have $\dim \mathcal{V}_0 \leqslant \operatorname{codim}(\mathcal{X}_{\text{det}} + \mathcal{X}_{\text{stab}}) + \dim \mathcal{X}_{\text{det}}$, and we conclude that the compensator constructed above leads to internal stability as well. ■

The proof is constructive once the pairs $(\mathcal{T}_0, \mathcal{V}_0)$ and $(\mathcal{T}, \mathcal{V})$ are given. We shall now proceed to discuss how the existence of these pairs can be verified by an algorithm that will also construct such pairs, if they exist.

6. A VERIFICATION ALGORITHM

It may not seem easy to verify the conditions of Theorem 5.3, because they are stated in terms of two (C, A, B)-pairs, which gives us four variable subspaces. Without much effort, one can see that $\mathcal{T}_0$ can always be replaced by $\mathcal{X}_{\text{det}}$ and $\mathcal{T}$ by $\mathcal{T}_g^*(\operatorname{im} E)$, but that still leaves us with two variable subspaces. It is possible to express the conditions in terms of $\mathcal{V}_0$ (as in [18] and [19]), but concentrating on $\mathcal{V}$ will lead to a result that is more attractive from a numerical point of view. Before we come to this, some preliminary work is needed.

LEMMA 6.1. *Let $\mathcal{V}$ be an (A, B)-invariant subspace, and let $\mathcal{R}$ be defined by $\mathcal{R} = \langle A + BF | \operatorname{im} B \cap \mathcal{V} \rangle$ $(F \in \mathbf{F}(\mathcal{V}))$. (This defines $\mathcal{R}$ uniquely: see Lemma 2.2.) If $\mathcal{V}_1$ is an (A, B)-invariant subspace such that $\mathcal{R} \subset \mathcal{V}_1 \subset \mathcal{V}$, then $\mathcal{V}_1$ is $(A + BF)$-invariant for all $F \in \mathbf{F}(\mathcal{V})$.*

Proof. Take $F \in \mathbf{F}(\mathcal{V})$, and $F_0 \in \mathbf{F}(\mathcal{V}_1)$. We have $(A + BF)\mathcal{V}_1 \subset (A + BF)\mathcal{V} \subset \mathcal{V}$, but also $(A + BF)\mathcal{V}_1 \subset (A + BF_0)\mathcal{V}_1 + B(F - F_0)\mathcal{V}_1 \subset \mathcal{V}_1 + \operatorname{im} B$. Hence $(A + BF)\mathcal{V}_1 \subset \mathcal{V} \cap (\mathcal{V}_1 + \operatorname{im} B) \subset \mathcal{V}_1 + \mathcal{R} \subset \mathcal{V}_1$. ■

LEMMA 6.2. *Let $\mathcal{V}_0$ be an (A, B)-invariant subspace contained in a subspace $\mathcal{K}$. The set of all (A, B)-invariant subspaces $\mathcal{V}$ contained in $\mathcal{K}$ and containing $\mathcal{V}_0$, that are such that (A, B) is stabilizable between $\mathcal{V}_0$ and $\mathcal{V}$, contains a unique maximal element, which is given by $\mathcal{V}_0 + \mathcal{V}_g^*(\mathcal{K})$.*

Proof. It follows immediately from Corollary 4.7 that (A, B) is stabilizable between $\mathcal{V}_0$ and $\mathcal{V}_0 + \mathcal{V}_g^*(\mathcal{K})$. Conversely, let $\mathcal{V}$ be an (A, B)-invariant subspace with $\mathcal{V}_0 \subset \mathcal{V} \subset \mathcal{K}$, such that (A, B) is stabilizable between $\mathcal{V}_0$ and $\mathcal{V}$. Then there exists an $F \in \mathbf{F}(\mathcal{V}_0) \cap \mathbf{F}(\mathcal{V}) \cap \mathbf{F}(\mathcal{V}^*(\mathcal{K}))$ such that $\sigma(A + BF: \mathcal{V}/\mathcal{V}_0) \subset \mathbf{C}_g$. By Lemma 6.1, we automatically have $F \in \mathbf{F}(\mathcal{V}_g^*(\mathcal{K}))$ as well. Now, on the one hand,

$$\begin{aligned}\sigma\big(A + BF: \mathcal{V}/\{\mathcal{V} \cap [\mathcal{V}_g^*(\mathcal{K}) + \mathcal{V}_0]\}\big) \\ &= \sigma\big(A + BF: [\mathcal{V}_g^*(\mathcal{K}) + \mathcal{V}]/[\mathcal{V}_g^*(\mathcal{K}) + \mathcal{V}_0]\big) \\ &\subset \sigma\big(A + BF: \mathcal{V}^*(\mathcal{K})/\mathcal{V}_g^*(\mathcal{K})\big) \subset \mathbf{C}_b,\end{aligned}$$

but on the other hand,

$$\sigma\big(A + BF: \mathcal{V}/\{\mathcal{V} \cap [\mathcal{V}_g^*(\mathcal{K}) + \mathcal{V}_0]\}\big) \subset \sigma\big(A + BF: \mathcal{V}/\mathcal{V}_0\big) \subset \mathbf{C}_g. \quad (6.2)$$

It follows that

$$\sigma\big(A + BF: \mathcal{V}/\{\mathcal{V} \cap [\mathcal{V}_g^*(\mathcal{K}) + \mathcal{V}_0]\}\big) = \varnothing, \quad (6.3)$$

or $\mathcal{V} \subset \mathcal{V}_g^*(\mathcal{K}) + \mathcal{V}_0$. ∎

We can now reformulate Theorem 5.3 as follows.

THEOREM 6.3. *The algebraic regulator problem for the system* (2.1–3) *is solvable if and only if there exists an (A, B)-invariant subspace $\mathcal{V}$ such that*

$$\mathcal{V} \subset \ker D, \quad (6.4)$$

$$\mathcal{V} + \mathcal{X}_{\text{stab}} = \mathcal{X}, \quad (6.5)$$

$$\mathcal{V} \cap (\mathcal{X}_{\text{det}} + \mathcal{X}_{\text{stab}}) = \mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D), \quad (6.6)$$

$$\mathcal{T}_g^*(\operatorname{im} E) \subset \mathcal{V}. \quad (6.7)$$

Proof. To prove the necessity, we assume that the conditions of Theorem 5.3 hold. So we have two compatible (C, A, B)-pairs $(\mathcal{T}_0, \mathcal{V}_0)$ and $(\mathcal{T}, \mathcal{V})$ satisfying (i)–(v). We shall show that $\mathcal{V}_0 + \mathcal{V}_g^*(\ker D)$ satisfies the conditions (6.4–7). From (i) and (ii) we immediately have (6.4), (6.5) follows directly from (i) and (iii) with use of Lemma 2.3, and (6.7) is obtained from (ii) and (iii) by an application of Lemma 6.2. Finally, the obvious fact that $\mathcal{V}_g^*(\ker D) \subset \mathcal{X}_{\text{det}} + \mathcal{X}_{\text{stab}}$ entails, by condition (v),

$$\left(\mathcal{V}_0 + \mathcal{V}_g^*(\ker D)\right) \cap \left(\mathcal{X}_{\text{det}} + \mathcal{X}_{\text{stab}}\right) = \mathcal{V}_0 \cap \left(\mathcal{X}_{\text{det}} + \mathcal{X}_{\text{stab}}\right) + \mathcal{V}_g^*(\ker D)$$
$$= \mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D). \tag{6.8}$$

For the sufficiency, we note that $\mathcal{X}_{\text{det}} \subset \mathcal{T}_g^*(\operatorname{im} E) \subset \mathcal{V} \subset \ker D$, and we consider the chain of (A, B)-invariant subspaces $\{0\} \subset \mathcal{X}_{\text{det}} \subset \mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D) \subset \mathcal{V} \subset \mathcal{X}$. Corollary 4.7 shows that (A, B) is stabilizable between $\mathcal{X}_{\text{det}}$ and $\mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D)$, and (6.5) shows that (A, B) is also stabilizable between $\mathcal{V}$ and $\mathcal{X}$. By Lemma 5.2, there exists an $F \in \mathbf{F}(\mathcal{X}_{\text{det}}) \cap \mathbf{F}(\mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D)) \cap \mathbf{F}(\mathcal{V})$ such that $\ker F \supset \mathcal{X}_{\text{det}}$, $\sigma(A + BF : (\mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D))/\mathcal{X}_{\text{det}}) \subset \mathbf{C}_g$, and $\sigma(A + BF : \mathcal{X}/\mathcal{V}) \subset \mathbf{C}_g$. Now, define $\mathcal{V}_0$ by

$$\mathcal{V}_0 = \mathcal{X}_b(A + BF). \tag{6.9}$$

It is clear that $\mathcal{X}_{\text{det}} \subset \mathcal{V}_0 \subset \mathcal{V}$, and also that $\mathcal{V}_0 \cap (\mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D)) = \mathcal{X}_{\text{det}}$. Using (6.6) we see that condition (v) of Theorem 5.3 is satisfied. We also see that (iii) holds. The other conditions are easily verified, if we define $\mathcal{T}_0 = \mathcal{X}_{\text{det}}$ and $\mathcal{T} = \mathcal{T}_g^*(\operatorname{im} E)$. ■

The following slight variation of this result will be useful.

COROLLARY 6.4. *The algebraic regulator problem for the system* (2.1–3) *is solvable if and only if*

$$\mathcal{V}^*(\ker D) + \mathcal{X}_{\text{stab}} = \mathcal{X} \tag{6.10}$$

and there exists an (A, B)*-invariant subspace* $\mathcal{V}$ *such that*

$$\mathcal{V} \subset \mathcal{V}^*(\ker D), \tag{6.11}$$

$$\mathcal{V} + \{\mathcal{X}_{\text{det}} + [\mathcal{V}^*(\ker D) \cap \mathcal{X}_{\text{stab}}]\} = \mathcal{V}^*(\ker D), \tag{6.12}$$

$$\mathcal{V} \cap \{\mathcal{X}_{\text{det}} + [\mathcal{V}^*(\ker D) \cap \mathcal{X}_{\text{stab}}]\} = \mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D), \tag{6.13}$$

$$\mathcal{T}_g^*(\operatorname{im} E) \subset \mathcal{V}. \tag{6.14}$$

Proof. Necessity: (6.10) follows from (6.4) and (6.5), (6.12) is obtained by intersecting both sides of the equality in (6.5) with $\mathcal{V}^*(\ker D)$, and (6.13) is obtained in the same way from (6.6). Sufficiency: For (6.5), add $\mathcal{X}_{\text{stab}}$ on both sides of (6.12) and use (6.10). Note that $\mathcal{X}_{\text{det}} \subset \mathcal{V}$ by (6.13) [or (6.14)], and consequently

$$\mathcal{V} \cap \{\mathcal{X}_{\text{det}} + [\mathcal{V}^*(\ker D) \cap \mathcal{X}_{\text{stab}}]\} = \mathcal{X}_{\text{det}} + (\mathcal{V} \cap \mathcal{X}_{\text{stab}})$$
$$= \mathcal{V} \cap (\mathcal{X}_{\text{det}} + \mathcal{X}_{\text{stab}}). \quad (6.15)$$

Now use (6.13) again to obtain (6.6). ■

We see from Theorem 6.3 that the subspace $\mathcal{V}$ that we are looking for must be in between $\mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D)$ and $\mathcal{V}^*(\ker D)$, and the advantage of the corollary is that the crucial conditions (6.12–13) are formulated in terms of these subspaces and of another subspace that is in between the two, $\mathcal{X}_{\text{det}} + (\mathcal{V}^*(\ker D) \cap \mathcal{X}_{\text{stab}})$. Pictorially, the situation we are trying to establish looks like Figure 6.

The important point to note here is that we are talking about (A, B)-invariant subspaces that all contain the subspace

$$\mathcal{R}^*(\ker D) := \langle A + BF | \operatorname{im} B \cap \mathcal{V}^*(\ker D)\rangle \qquad [F \in \mathbf{F}(\mathcal{V}^*(\ker D))] \quad (6.16)$$

and which are therefore, by Lemma 6.1, all invariant for each $F \in$

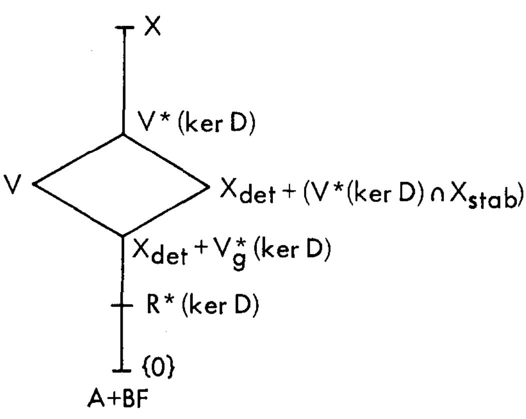

FIG. 6. The splitoff property.

$\mathbf{F}(\mathcal{V}^*(\ker D))$. This means that we can pick any $F \in \mathbf{F}(\mathcal{V}^*(\ker D))$ and see if a subspace $\mathcal{V}$ can be "split off" as depicted in Figure 6. This comes down to requiring that the subspace

$$[\mathcal{X}_{\text{det}} + (\mathcal{V}^*(\ker D) \cap \mathcal{X}_{\text{stab}})]/[\mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D)] \tag{6.17}$$

must decompose the quotient space $\mathcal{V}^*(\ker D)/[\mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D)]$ with respect to the mapping induced by $A + BF$ on this space. This is well known to be equivalent to a linear matrix equation (see for instance [2, p. 21]). The conclusion that we have now reached should be compared to Theorems 7.3 and 7.4 in [2]. In particular, the problem is trivial under the *minimum-phase condition* $\mathcal{V}_g^*(\ker D) = \mathcal{V}^*(\ker D) \cap \mathcal{X}_{\text{stab}}$: in this case, the only solution of (6.11–13) is $\mathcal{V} = \mathcal{V}^*(\ker D)$. (This condition is often assumed in classical control theory, though not quite in this formulation.)

To obtain a computational criterion, we may proceed as follows. Noting that it is necessary that $\mathcal{X}_{\text{det}} \subset \mathcal{V}^*(\ker D)$, we may set up a basis for $\mathcal{X}$ that is adapted to the chain of subspaces $\{0\} \subset \mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D) \subset \mathcal{X}_{\text{det}} + [\mathcal{V}^*(\ker D) \cap \mathcal{X}_{\text{stab}}] \subset \mathcal{V}^*(\ker D) \subset \mathcal{X}$. Next, we form block matrix representations for the relevant mappings and subspaces. We get

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix},$$

$$B = \begin{pmatrix} B_1 \\ 0 \\ 0 \\ B_4 \end{pmatrix}, \qquad \mathcal{T}_g^*(\operatorname{im} E) = \operatorname{sp}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix}. \tag{6.18}$$

Here, the fact that $A_{21} = 0$ and $A_{31} = 0$ is explained by noting that

$$\begin{aligned} A(\mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D)) \cap \mathcal{V}^*(\ker D) & \\ &\subset (\mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D) + \operatorname{im} B) \cap \mathcal{V}^*(\ker D) \\ &\subset \mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D) + (\operatorname{im} B \cap \mathcal{V}^*(\ker D)) \\ &= \mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D). \end{aligned} \tag{6.19}$$

The same explanation goes for $A_{32} = 0$. We have used the fact that $\operatorname{im} B \cap \mathcal{V}^*(\ker D) \subset \mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D)$, which also entails $B_2 = 0$ and $B_3 = 0$.

A subspace $\mathcal{V}$ satisfies (6.11–13) if and only if it can be represented, with respect to the selected basis, in the following way:

$$\mathcal{V} = \operatorname{sp}\begin{pmatrix} I & 0 \\ 0 & X \\ 0 & I \\ 0 & 0 \end{pmatrix}, \tag{6.20}$$

where X may be any matrix of suitable size. Such a subspace is (A, B)-invariant, by Lemma 2.1, if and only if there exist matrices Q and R such that

$$\begin{pmatrix} A_{11} & A_{12}X + A_{13} \\ 0 & A_{22}X + A_{23} \\ 0 & A_{33} \\ A_{41} & A_{42}X + A_{43} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & X \\ 0 & I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \\ 0 \\ B_4 \end{pmatrix}\begin{pmatrix} R_1 & R_2 \end{pmatrix}. \tag{6.21}$$

By elementary calculations, and using the fact that there exist, by the (A, B)-invariance of $\mathcal{V}^*(\ker D)$, matrices F_i such that $A_{4i} + B_4F_i = 0$ ($i = 1,\dots,3$), we find that such matrices Q and R exist if and only if

$$A_{22}X + A_{23} = XA_{33}. \tag{6.22}$$

(This is, of course, Sylvester's equation [2, p. 21].) Furthermore, the condition (6.14) holds if and only if there exists a matrix S such that

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & X \\ 0 & I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}. \tag{6.23}$$

This is true if and only if $T_4 = 0$ and

$$T_2 = XT_3. \tag{6.24}$$

Our conclusion is as follows.

COROLLARY 6.5. *The algebraic regulator problem for the system* (2.1–3) *is solvable if and only if the following conditions hold*:

$$\mathcal{V}^*(\ker D) + \mathcal{X}_{\text{stab}} = \mathcal{X}, \tag{6.25}$$

$$\mathcal{T}_g^*(\operatorname{im} E) \subset \mathcal{V}^*(\ker D), \tag{6.26}$$

and there exists a matrix X satisfying

$$XA_{33} - A_{22}X = A_{23}, \tag{6.27}$$

$$XT_3 = T_2, \tag{6.28}$$

where the A- and T-matrices are defined as in (6.18).

For any $F \in \mathbf{F}(\mathcal{V}^*(\ker D))$, A_{33} is the matrix of

$$A + BF : \mathcal{V}^*(\ker D)/\{\mathcal{X}_{\text{det}} + [\mathcal{V}^*(\ker D) \cap \mathcal{X}_{\text{stab}}]\}, \tag{6.29}$$

and A_{22} is the matrix of

$$A + BF : \{\mathcal{X}_{\text{det}} + [\mathcal{V}^*(\ker D) \cap \mathcal{X}_{\text{stab}}]\}/[\mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D)]. \tag{6.30}$$

Under the conditions (6.25–26), the mapping in (6.29) is similar to $A : \mathcal{X}/(\mathcal{X}_{\text{det}} + \mathcal{X}_{\text{stab}})$, and so we can say that A_{33} represents the *signal dynamics*. In view of the interpretations of Section 3 and of [2, Section 5.5], the eigenvalues of the mapping in (6.30) may be identified as the relevant *unstable plant zeros*. In particular, since we know that the equation (6.27) has a unique solution if and only if the matrices A_{22} and A_{33} have no eigenvalues in common [26, p. 225], we can say that a sufficient condition for (6.27) to be solvable is that the signal poles and the relevant unstable plant zeros are distinct.

It should be emphasized that several numerical techniques are available to verify the conditions (6.25–28). The computation of $\mathcal{V}^*(\ker D)$ and related subspaces and mappings is discussed from the numerical point of view in [20–22]. The equation (6.27) can be solved efficiently, at least in the case where the eigenvalues of A_{22} and A_{33} are distinct, by the method of [23]. Note that the size of A_{22} is the number of unstable plant zeros, whereas the size of A_{33} is the number of signal poles, and both numbers will be moderate in very many situations. Finally, if (6.27) has a unique solution, then (6.28) is just a matter of checking. All this gives hope that the solution provided by Corollary 6.5 will be a good foundation for developing numerical software for general regulator problems.

7. THE INTERNAL MODEL

Francis [7] proved that, in the special case where the output is the same as the observation ($C = D$), any compensator that solves the algebraic regulator problem must contain a copy of the signal dynamics, the so-called "internal model." A similar result was derived by Bengtsson [8] in a frequency-domain setting. Another form of the internal-model principle, which involves a certain reduplication of signal dynamics, can be derived from strong robustness requirements: see [2, Chapter 8]. Below, we shall show how the internal model can be obtained from the setup presented here. Our result is slightly more general than that of Francis.

PROPOSITION 7.1. *Suppose that the compensator* (2.4–5) *provides a solution to the algebraic regulator problem for the system* (2.1–3), *in which* $\ker D \subset \ker C$. *Then there exists an A_c-invariant subspace $\mathcal{W}_0$ of $\mathcal{W}$ such that $A_c : \mathcal{W}_0$ is similar to $A : \mathcal{X}/(\mathcal{X}_{\text{det}} + \mathcal{X}_{\text{stab}})$.*

Proof. Write $\mathcal{T}_0 = Q^{-1}\mathcal{X}_b^e(A_e)$, $\mathcal{V}_0 = P\mathcal{X}_b^e(A_e)$. We first show that $\mathcal{T}_0 = \mathcal{X}_{\text{det}}$. Being a (C, A)-invariant subspace in $\mathcal{V}_0 \subset \ker D \subset \ker C$, $\mathcal{T}_0$ must in fact be A-invariant. So we have $\mathcal{T}_0 \subset \langle \ker C | A \rangle$. It is easily checked that $Q(\langle \ker C | A \rangle)$ is A_e-invariant and that Q intertwines $A : \langle \ker C | A \rangle$ and $A_e : Q(\langle \ker C | A \rangle)$. From this, it follows that $\mathcal{T}_0 = \mathcal{X}_{\text{det}}$.

From Proposition 4.8, Lemma 2.3, and the formulation of internal stability in (3.7), we see that

$$\dim \mathcal{V}_0 = \dim \mathcal{X}_b^e(A_e). \tag{7.1}$$

This implies that there exists a mapping $L : \mathcal{V}_0 \to \mathcal{W}$ such that

$$\mathcal{X}_b^e(A_e) = \left\{ \begin{pmatrix} x \\ Lx \end{pmatrix} \middle| x \in \mathcal{V}_0 \right\}. \tag{7.2}$$

Because $\mathcal{V}_0 \subset \ker D \subset \ker C$, we have, for $x \in \mathcal{V}_0$,

$$A_e \begin{pmatrix} x \\ Lx \end{pmatrix} = \begin{pmatrix} (A + BF_c L)x \\ A_c Lx \end{pmatrix} \in \mathcal{X}_b^e(A_e). \tag{7.3}$$

Writing $F_0 := F_c L$, we find that $(A + BF_0)\mathcal{V}_0 \subset \mathcal{V}_0$ and $A_c Lx = L(A + BF_0)x$ for $x \in \mathcal{V}_0$. This means that $\mathcal{W}_0 := \operatorname{im} L$ is A_c-invariant, and that $A_c : \mathcal{W}$ is similar to $A + BF_0 : \mathcal{V}_0/(\ker L)$. Because $\ker L = Q^{-1}\mathcal{X}_b^e(A_e) = \mathcal{X}_{\text{det}}$, it remains to show that $A + BF_0 : \mathcal{V}_0/\mathcal{X}_{\text{det}}$ is similar to $A : \mathcal{X}/(\mathcal{X}_{\text{det}} + \mathcal{X}_{\text{stab}})$. By Proposition 4.8 and Lemma 2.3, we have

$$\begin{aligned} A + BF_0 : \mathcal{V}_0/\mathcal{X}_{\text{det}} &\cong A + BF_0 : \mathcal{V}_0/[\mathcal{V}_0 \cap (\mathcal{X}_{\text{det}} + \mathcal{X}_{\text{stab}})] \\ &\cong A + BF_0 : [\mathcal{V}_0 + (\mathcal{X}_{\text{det}} + \mathcal{X}_{\text{stab}})]/(\mathcal{X}_{\text{det}} + \mathcal{X}_{\text{stab}}) \\ &\cong A : \mathcal{X}/(\mathcal{X}_{\text{det}} + \mathcal{X}_{\text{stab}}). \end{aligned} \tag{7.4}$$

This concludes the proof. ■

The internal-model principle does not have to hold if there are observations available that are independent of the output ($\ker D \not\subset \ker C$). Indeed,

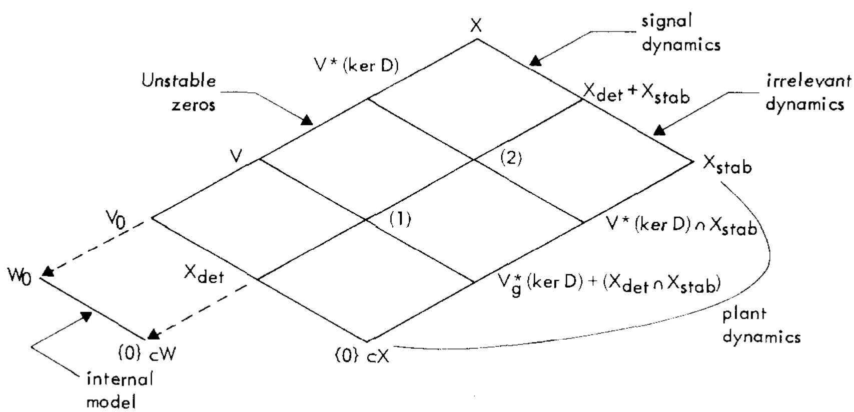

FIG. 7. Structure of the algebraic regulator problem: (1) $\mathcal{X}_{\text{det}} + \mathcal{V}_g^*(\ker D)$, (2) $\mathcal{X}_{\text{det}} + (\mathcal{V}^*(\ker D) \cap \mathcal{X}_{\text{stab}})$.

the disturbance-decoupling problem may be solvable in nontrivial cases, and then any relation between the compensator dynamics and the disturbance dynamics is quite effectively precluded, since we did not make any assumption on the dynamics of the disturbance entering through E. The point is, of course, that the availability of observations independent of the output allows for a certain freedom of design, which may be used to advantage. The often used assumption $C = D$ is to be considered as a nontrivial specialization. Let us conclude by showing pictorially, in Figure 7, how the internal model fits into the structure discussed in previous sections. In this picture, only the presence of the internal model depends on the assumption $\ker D \subset \ker C$; all the rest holds in general.

8. CONCLUSIONS

We have been able to solve a general version of the algebraic regulator problem, requiring output stability, internal stability, and disturbance decoupling as well. The basic Theorem 5.3 has been derived in a quite straightforward way, using material that is essentially elementary, as it is also likely to be useful for the analysis of other feedback design problems. Among this material, especially useful are Lemma 4.1, which gives the connection between closed-loop invariant subspaces and (C, A, B)-pairs, and Lemma 4.4, which adds the stability aspects to this connection. On the constructive side, the versatile compensator construction of Lemma 5.1 is important, and the "paste-together" Lemma 5.2 comes in handy. The main drawback of the results that we get from this type of analysis, like Theorem 5.3, is that the solvability condition involves the existence of a number of (C, A, B)-pairs having certain properties, so that it remains to be seen how this condition is going to be verified. For some problems, it is possible to have canonical choices for the (C, A, B)-pairs in terms of computable subspaces like $\mathcal{X}_{\mathrm{det}}$, $\mathcal{T}_g^*(\mathrm{im}\, E)$, etc. (Examples of this are the disturbance decoupling problem with stability [27, 28, 19], which is obtained as a special case of the problem treated here by taking $\mathcal{X}_{\mathrm{stab}} = \mathcal{X}$, or the regulator problem under the minimum-phase condition.) For the general problem in hand, this turned out to be not completely possible. It was possible, however, to select one pivot subspace in which the solvability condition could be expressed (Theorem 6.3), and to derive a computational criterion for this subspace (Corollary 6.5) which also had a geometric interpretation as a decomposability condition. In this way, we obtained a fully effective solution.

Life was also made somewhat easier by the use of the subspaces $\mathcal{X}_{\mathrm{stab}}$ and $\mathcal{X}_{\mathrm{det}}$ rather than the subspaces $\langle A|\mathrm{im}\, B\rangle$ and $\langle \ker C|A\rangle$ which were employed in [1] and [2]. There is a clear parallel here with the recent trend in transfer-matrix analysis to the use of factorizations in stable rationals instead of polynomials (see for instance [12] and [14]).

This paper started out with an engineering motivation, and therefore it is perhaps proper to give an assessment of the value of the results in engineering terms. Since we took just a few aspects from the multifaceted problem of control-system design and pursued these only, while ignoring all other aspects, it must be said that these results are of little immediate value. One particular problem is that we have assumed that the parameters of the system to be controlled are known exactly, which is never true in real life. The assumption would be justified, though, if we knew that the performance of a controller designed for a given system would not deteriorate badly if this system were replaced by another one which was "close" to it. To consider this type of question, it is obviously necessary to introduce concepts of "closeness" and to study, let us say, topological properties. There is no disputing that "natural" topologies defined separately on the mappings that make up the state-space description are completely unsatisfactory in the context of control systems. Instead, one should look for topologies defined in terms of transfer matrices of the form $C(sI-A)^{-1}B$.

This being true, why then didn't we try to solve the problem from beginning to end in transfer-matrix terms? It is safe to say that, as long as the algebra remains fairly simple, the use of the transfer-function terminology provides a natural topological background which explains much of the famous engineering "feel" for control-system design. But if one wants to solve sharply defined problems which require an elaborate algebraic treatment, the use of this terminology no longer guarantees an easy linkup with topology, and one has to take this up as a separate subject. If this is necessary anyway, the option of using the state-space description becomes prominent again. In the fifties, Bellman and Kalman reemphasized the state-space method for various reasons, such as numerical advantages; but an important point also was the mathematical transparency that can be obtained from this approach. It is hoped that the present paper supports the contention that this argument is still valid.

APPENDIX

The purpose of this appendix is to prove that the algebraic regulator problem in our formulation is a strict generalization of the regulator problem with internal stability as studied in [1] (also [2, Chapter 7]). In fact, the RPIS is obtained from the problem solved here by setting $E=0$ in (2.1).

It is convenient to use a reformulation of Theorem 5.3 in terms of the subspace $\mathcal{V}_0$ (rather than $\mathcal{V}$, as was done in Theorem 6.3). The corresponding statement is as follows.

LEMMA A.1. *The algebraic regulator problem for the system* (2.1–3) *is solvable if and only if there exists an* (A, B)*-invariant subspace* $\mathcal{V}_0$ *such that*

(i) $\mathcal{V}_0 \subset \ker D$
(ii) $\mathcal{T}_g^*(\text{im } E) \subset \mathcal{V}_0 + \mathcal{V}_g^*(\text{ker}) D)$
(iii) $\mathcal{V}_0 + \mathcal{X}_{\text{stab}} = \mathcal{X}$
(iv) $\mathcal{X}_{\text{det}} = \mathcal{V}_0 \cap (\mathcal{X}_{\text{det}} + \mathcal{X}_{\text{stab}})$.

The proof requires no new techniques; see [18]. We now specialize to the case $E = 0$.

PROPOSITION A.2. *Consider the system* (2.1–3) *with E set equal to zero. The algebraic regulator problem is solvable if and only if the* RPIS *is solvable in the sense of* [1], *i.e., if there exists a feedback mapping* $F: \mathcal{X} \to \mathcal{U}$ *such that*

$$\ker F \supset \langle \ker C | A \rangle, \tag{A.1}$$

$$\mathcal{X}_b(A + BF) \cap (\langle A | \text{im } B \rangle + \langle \ker C | A \rangle) \subset \langle \ker C | A \rangle, \tag{A.2}$$

$$\mathcal{X}_b(A + BF) \subset \ker D. \tag{A.3}$$

Proof. First assume that there exists an F such that (A.1–3) is true, and write $\mathcal{V}_0 = \mathcal{X}_b(A + BF)$. We shall show that $\mathcal{V}_0$ satisfies the conditions of Lemma A.1. Note that condition (ii) of this lemma becomes subsumed under condition (iv) in the special case $E = 0$. It is clear that conditions (i) and (iii) hold [$\mathcal{X}_b(A + BF)$ is outer-stabilizable, of course]. Also, it follows from (A.1) that $\langle \ker C | A \rangle$ is $(A + BF)$-invariant, and that $\mathcal{X}_{\text{det}} \subset \mathcal{X}_b(A + BF)$. So we get $\mathcal{X}_{\text{det}} \subset \mathcal{V}_0 \cap (\mathcal{X}_{\text{det}} + \mathcal{X}_{\text{stab}})$. To prove the reverse inclusion, note that (A.2) is equivalent to

$$\sigma(A + BF: (\langle A | \text{im } B \rangle + \langle \ker C | A \rangle)/\langle \ker C | A \rangle) \subset \mathbb{C}_g$$

$$\leftrightarrow \quad \sigma(A + BF: \langle A | \text{im } B \rangle / (\langle A | \text{im } B \rangle \cap \langle \ker C | A \rangle)) \subset \mathbb{C}_g$$

$$\leftrightarrow \quad \mathcal{X}_b(A + BF) \cap \langle A | \text{im } B \rangle \subset \langle A | \text{im } B \rangle \cap \langle \ker C | A \rangle$$

$$\leftrightarrow \quad \mathcal{X}_b(A + BF) \cap \langle A | \text{im } B \rangle \subset \langle \ker C | A \rangle. \tag{A.4}$$

Because $\mathcal{X}_b(A+BF)\cap\langle\ker C|A\rangle=\mathcal{X}_{\det}$, this is equivalent to

$$\mathcal{X}_b(A+BF)\cap\langle A|\operatorname{im} B\rangle\subset\mathcal{X}_{\det}. \tag{A.5}$$

The subspace $\mathcal{X}_{\text{stab}}$ is the same for the pair $(A+BF, B)$ as it is for the pair (A, B), so we have

$$\begin{aligned}\mathcal{X}_{\text{stab}} &= \mathcal{X}_g(A+BF)+\langle A|\operatorname{im} B\rangle\\ &= \mathcal{X}_g(A+BF)+\left(\mathcal{X}_b(A+BF)\cap\langle A|\operatorname{im} B\rangle\right).\end{aligned} \tag{A.6}$$

Intersecting the extremes of (A.6) with $\mathcal{X}_b(A+BF)$, we obtain

$$\mathcal{X}_b(A+BF)\cap\mathcal{X}_{\text{stab}}=\mathcal{X}_b(A+BF)\cap\langle A|\operatorname{im} B\rangle. \tag{A.7}$$

The conclusion from (A.5) and (A.7) is

$$\mathcal{V}_0\cap\mathcal{X}_{\text{stab}}\subset\mathcal{X}_{\det}. \tag{A.8}$$

We already proved that $\mathcal{X}_{\det}\subset\mathcal{V}_0$, and under this condition (A.8) is equivalent to

$$\mathcal{V}_0\cap\left(\mathcal{X}_{\det}+\mathcal{X}_{\text{stab}}\right)\subset\mathcal{X}_{\det}, \tag{A.9}$$

which is what we wanted to prove.

Conversely, let us suppose that there exists an (A, B)-invariant subspace $\mathcal{V}_0$ such that conditions (i)–(iv) of lemma A.1 hold. Then we have to construct an F satisfying (A.1–3). It follows from condition (iv), by intersection of both sides of the equality with $\mathcal{X}_g(A)$, that

$$\mathcal{V}_0\cap\mathcal{X}_g(A)=\{0\}. \tag{A.10}$$

It is clear from this that we can define F on $\mathcal{V}_0\oplus\mathcal{X}_g(A)$ in such a way that $\ker F\supset\mathcal{X}_g(A)$ and $(A+BF)\mathcal{V}_0\subset\mathcal{V}_0$. We can also arrange that $\ker F\subset\mathcal{X}_{\det}$, because $\mathcal{X}_{\det}\subset\mathcal{V}_0$ by condition (iv). It follows from condition (iii) that $\mathcal{V}_0\cap\mathcal{X}_g(A)$ is outer-stabilizable, and so Lemma 5.2 shows that F can be extended to a mapping defined on all of $\mathcal{X}$ in such a way that $\sigma(A+BF\colon\mathcal{X}/[\mathcal{V}_0\oplus\mathcal{X}_g(A)])\subset\mathbb{C}_g$. We then have $\mathcal{X}_b(A+BF)\subset\mathcal{V}_0\subset\ker D$, which satisfies (A.3). Also, $\langle\ker C|A\rangle=\mathcal{X}_{\det}\oplus[\langle\ker C|A\rangle\cap\mathcal{X}_g(A)]\subset\ker F$. Finally, it is seen from (A.4) and (A.7) that (A.2) is equivalent to

$$\mathcal{X}_b(A+BF)\cap\mathcal{X}_{\text{stab}}\subset\langle\ker C|A\rangle. \tag{A.11}$$

But this is immediate from condition (iv) and the fact that $\mathcal{X}_b(A+BF)\subset\mathcal{V}_0$. ∎

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